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ABNORMAL VOLTAGES IN TRANSFORMERS

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ABSTRACT OF PAPER

This paper deals with the electrical behavior of transformer windings when subjected to steep wave fronts and high-frequency wave trains. The dependance of the internal voltages produced, upon the distribution of capacity with the inductance of the winding is discussed.

Practical windings are divided into two general classes, one in which inductance and capacity are practically uniformly distributed, and the other in which the capacity is more or less concentrated at certain points, with relatively concentrated portions of inductance intervening.

Neglecting the effects of the high frequency dielectric losses in the insulation at high frequency, distinct mathematical analysis is given to these two classes of winding to determine the ratios of the internal voltages to the voltage of the external wave or wave train. The resulting internal voltage distributions are plotted for various frequencies, and curves are plotted for the relations of maximum internal voltages to frequency. These curves show that some frequencies are dangerous, while others are not, but it can not be said that one of these types of winding is better than the other from the standpoint of the possibility of excessive internal voltages.

The analysis is by no means complete, but an examination is made of the facts and fundamental principles involved which will enable us to insulate for and guard against excessive internal voltages in a more scientific manner.

FROM THE first use of transformers, the occurrence of excessive voltages between adjacent turns or sections of the same winding, as a result of switching, sparking discharges or lightning, forced itself upon our attention by its results in punctured insulation. In the early stages of development, the effort was only to insulate adjacent turns and coils from each other in a manner to give a satisfactory factor of safety for the calculated transformer voltages, and lightning arresters were developed for protection from excessive external voltages. It was found, however, that breaks between turns—usually the end turns of the winding—were altogether too frequent. It was then recognized that a sudden application of voltage, or sudden change of voltage, at the transformer terminals, would not instantly distribute itself throughout the winding, but was

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more or less concentrated at the first instant across the end turns. It was seen that this condition was not remedied by lightning arresters, and the use of extra end turn insulation came into vogue. Also the practise of connecting choke coils between the transformer winding and the line.

Some very lively discussions occurred as to the relative merits of external inductance and end-turn insulation. Recommendations were made as to the amount of inductance necessary, on the one hand, and, on the other, for the portion of the winding requiring extra insulation. But with increased experience it was found that the protection afforded by an external inductance in many cases did not eliminate the necessity for extra turn insulation. Also that with extra end turn insulation more frequent breaks occurred further in from the ends, and, in fact, the extra insulation on the end turns was itself occasionally subject to failure. To meet the requirements for safety, as indicated by experience, this insulation has been gradually increased, and gradually extended further from the ends, until at present many large high voltage transformers have what might be called re-enforced turn insulation throughout. Where a factor of four, for instance, applied to normal voltages to ground, gives ample strength for the major insulation between the high-voltage winding and the low-voltage winding and core, a factor of from 50 to 100 times as great is used in these transformers with respect to the insulation between turns. Thus, this insulation has a strength several hundred times as great as the normal ratio voltage between turns.

In this matter of internal insulation, experience has been a tedious and a costly teacher. When failure has occurred, the cause was often so uncertain as to leave doubts as to the merits of the case. There were so many causes to which the difficulty might be ascribed, such as dirt, moisture and faulty construction or abnormal operation, that a number of failures were required to convince the designers that more insulation was needed, or that the design should be modified. Moreover it has been necessary throughout for the designer to entertain considerations not only of safety, but also of economy. Up to the present, however, he has been handicapped in his efforts to properly adjust these contending considerations by the lack of adequate physical conceptions; first, of the nature and extent of the disturbances from which external protection cannot reasonably be expected; and second, of the electrical behavior of the windings themselves when subjected to these disturbances. These questions have been the subject of special investigation, with the result that a new epoch in transformer design has been initiated, based upon a better knowledge of the physics involved. The present paper deals only with the second question, no consideration being given to the severity of the disturbances which may occur in practise.

Our consideration of the nature of the effects produced will be based upon two classes of disturbances, which are typical; namely, the high-frequency wave train and the abrupt wave front. If we consider the effects of sheer wave fronts, or voltage changes of given amplitude which are absolutely sudden, and of wave trains which are sustained with uniform amplitude, we will have considered conditions which are worse in these respects than the worst which occur in practise. These considerations will, however, show us the nature of the results which may occur in practise, and mathematical discussion is more easily based upon these extreme conditions.

Transformer windings are ordinarily thought of merely as large inductances. In reality, they contain a rather large amount of capacity distributed in different ways depending upon the type and arrangement of the winding. For ordinary normal operating frequencies, in a steady state, the effect of capacity is negligible, and the winding acts like a simple concentrated inductance, with voltage uniformly distributed. At high frequencies, however, or when a sudden voltage is impressed, the effect of capacity in disturbing this voltage distribution becomes important. This is due to the fact that at high frequencies, conditions of resonance are reached for the various combinations of inductance and capacity. This action will be understood when we consider certain typical combinations.

Considering first the effect of an alternating voltage impressed upon an inductance and a capacity in parallel; the current taken by the capacity, with constant voltage, is directly proportional to the frequency, while the current taken by the inductance is inversely proportional thereto. The frequency at which these currents are equal is called the resonant frequency for this combination. The direction of these currents with respect to the external circuit, or source of impressed voltage, are opposed to each other, so that this combination takes no resultant current from the external circuit at its resonant frequency, however high the voltage may be. It therefore acts, under these conditions, like an open circuit.

At frequencies below the resonant frequency for this combination, the resultant current will be the excess of that taken by the inductance over that taken by the capacity, while at frequencies above the resonant frequency, the resultant current will be the excess of that taken by the capacity over that taken by the inductance. Thus, so far as the external circuit is concerned, this combination acts like an inductance at frequencies below its resonant frequency, and like a capacity at frequencies above its resonant frequency.

If, now, we have two parallel arrangements of inductance and capacity, in series with each other, the resonant frequency of the two arrangements being different, the action, at frequencies between these resonant frequencies, so far as the voltages across the individual arrangements and the current in the external circuit are concerned, is the same as with an inductance and a capacity in series.

Considering the effect of the impressed alternating voltage upon an inductance and a capacity in series, we find the same current in both, while the voltages across the inductance and the capacity are in opposition to each other. The voltage impressed upon the combination is the resultant, or the arithmetic difference between these two voltages. With constant current in this circuit, the voltage across the inductance is proportional to the frequency, while that across the capacity is inversely proportional to it. The resonant frequency is that frequency at which these voltages are equal to each other. With any finite voltage across the combination it is seen that the resonant frequency would result in infinite voltages across the separate elements of inductance and capacity, except for the effects of losses which exist within the inductance and capacity, which will not be described here. The current, however, which can be supplied by the generator or the line, is limited, and since the voltages across the inductance and the capacity are fixed by the current, the resultant of these two equal and opposite finite voltages would be zero. This combination, at its resonant frequency, acts therefore like a short circuit. At frequencies lower than the resonant frequency for this combination, the voltage across the capacity will be greater than that across the inductance, while at frequencies higher than the resonant frequency, the voltage across the inductance will be greater. This combination will therefore act like a capacity at frequencies below its resonant frequency, and like an inductance at frequencies above that frequency.

In various types of windings, as stated above, we find capacity distributed with the inductance in various ways. There is not only capacity to ground, as represented in transformers by the core and case, but also capacity between parts of the winding, and in transformers capacity to the opposite winding. The capacities between portions of the same winding are capacities in parallel with the inductances of those portions. Also the capacities to ground, in the case of a winding which possesses a fixed ground or definite neutral point, are in parallel with the inductances between the points where the capacities are located and the ground or neutral point. We have, therefore, various parallel combinations of inductance and capacity in series with various other such combinations, which gives opportunities for resonance and excessive internal voltages at various points within the winding, occurring respectively at different frequencies.

We will now consider the effects of a sudden or abrupt voltage impressed upon typical combinations of inductance and capacity. In the case of a simple capacity, the current at the first instant is limited only by the external or supply circuit. Since current cannot be built up instantly in supply circuits which we have to consider, on account of their inductance, the voltage across the condenser will start at zero, at the first instant, and as the condenser becomes charged it will build up to the full value, and current will cease. At the first instant it acts like a short circuit, but in its final state like an open circuit.

With a simple inductance, the action will be just the reverse of that with the capacity. The current, at the first instant, will be zero, with the full value of voltage, but ultimately the current is limited by the supply circuit only, and the voltage across the inductance is zero. At the first instant the inductance acts like an open circuit, but in its final state like a short circuit.

With an inductance and capacity in parallel, the combination acts like a short circuit, at the first instant, due to the presence of the condenser, and in its final state, it acts like a short circuit, due to the presence of the inductance. During the intermediate period a certain voltage will grow, and then disappear, due to the combined action of the capacity and the inductance, but this voltage will never reach the value which would appear with an open circuit, since current exists during this period in both the inductance and the capacity.

With an inductance and capacity in series, the combination acts like an open circuit, both at the first instant, because current cannot flow instantly through the inductance, and in its final state, because current cannot flow continuously through the capacity. At the first instant, the total voltage is across the inductance, while in the final condition, it is all across the capacity. During the interval between the first instant and the final condition, an oscillation takes place, with a maximum voltage across the inductance equal to the impressed voltage, and a maximum voltage across the capacity of double that value.

A combination such as is found in windings, as described above; namely, various parallel arrangements of inductance and capacity in series with other parallel arrangements, will act like a short circuit at the first instant, on account of the existence of the series of condensers across the entire combination. It will also act like a short circuit in its final state, on account of the series of inductances. During the intermediate period, the voltage across the combination will grow and then disappear in a manner similar to that mentioned above for a single inductance in parallel with a single capacity. If the various capacity units in this arrangement are in inverse proportion with the respective inductance units with which they are in parallel, the voltage will at all times be uniformly distributed throughout the inductance. That is, the voltage across the various inductance units will be proportional to the respective amounts of inductance, and in inverse proportion to the amounts of capacity. This result is produced with a current flowing through the series of inductances, with the same value in all, and another distinct current flowing through the series of condensers, with the same value in all.

If the capacities and inductances are not in the proportions specified above, the uniform distribution of voltage will not exist, but oscillatory disturbances will be set up which are similar in a general way to those produced by the capacity and inductance in series, due to the fact that, since currents cannot flow at the same time through the inductances with the same value in all and through the condensers with the same value in all, currents which flow through inductance in one part of the combination must flow through capacity in another part. The exact nature of the oscillation produced depends upon the particular combination of inductance and capacity which is found, and would need investigation for the particular case.

If consecutive voltages, or changes in voltage, be impressed upon the combination of inductance and capacity found in a winding, the resulting oscillations are superposed upon each other. It is clear, therefore, that excessive voltages may be built up by a series of such voltage changes or wave fronts, occurring at intervals corresponding to a resonant frequency of the particular combination of inductance and capacity. Such a series of waves would, in fact, constitute a wave train of the resonant frequency for this combination.

While the distribution of capacity with inductance in transformer windings is ordinarily too complicated to be accurately expressed in simple terms for mathematical analysis, yet windings may be divided into two general classes, which are roughly represented by simple typical arrangements, and an investigation of the behavior of these simplified arrangements of inductance and capacity will give us a very satisfactory conception of the behavior of the two types of windings. We will now proceed to investigate in some detail these two classes of windings, which are:

1. Windings in which inductance and capacity are practically uniformly distributed.

2. Windings in which the capacity is more or less concentrated and localized at certain places in the winding, the intervening portions of the winding constituting relatively concentrated inductances.

There is no definite line of division between these two classes, since, if the individual portions of inductance and capacity are relatively small, and the frequency relatively low, the latter type of winding will act like the former type, in accordance with the principle by which a telephone line loaded with inductance coils at sufficiently frequent intervals acts like a line with uniformly distributed inductance with respect to telephone currents, which have a wave length sufficiently great that several of the loading coils occur within the length of a half wave in the line. The classification as to behavior of any given winding depends, therefore, upon the frequency of the disturbance that is considered.

A winding consisting of a single cylindrical layer is the simplest example of the first type mentioned above, while the second is represented by a winding consisting of groups of pancake coils interlaced with the coils of a low-voltage winding, which will here be considered as ground. We will first consider the behavior of:

Windings with Inductance and Capacity Uniformly Distributed. A wave arriving from the line at the terminals of a winding of this type will be partially reflected and partially propagated into and through the winding. It will produce no internal oscillation, but will merely pass along the winding, with velocity reduced and voltage increased from those found in the line. The tendency of both of these effects is to produce a steeper wave front in the winding than that found in the line, with correspondingly large transient voltages between turns. If the wave front is steep, however, there is, on the other hand, a very important opposite tendency to reduce the voltages between turns due to the capacity between turns.

If we consider, for instance, the effect of a sheer wave front arriving at the terminals of such a winding, the distribution of voltage, at the first instant, will depend upon the distribution of capacity from the first turn to ground. Referring to Fig. 1, which represents the type of winding considered, it is seen that this capacity is represented by a series of condensers between adjacent turns, with the capacity of each turn to ground shunting the part of the system beyond that turn.

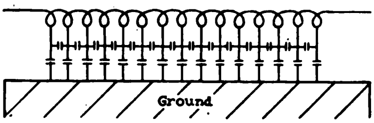


FIG. 1
Diagram of capacity and inductance in winding with uniformly distributed capacity

This combination of capacities is the same as that investigated by Mr. F. W. Peek, in his paper on "Electrical Characteristics of the Suspension Insulator," TRANSACTIONS of A.I.E.E., Vol. 31, Part I, pages 907-930. If the capacities between turns are large as compared with the capacities of the individual turns to ground, and this is the condition found in practise, the total voltage, at the first instant, is distributed by the action of the system of condensers over a considerable number of turns. Although the maximum voltage will be found on the first turn, this will be a small percentage of the total voltage, and smaller, the larger the capacities from turn to turn. This indicates a disadvantage of extra end-turn insulation, which reduces the capacity between turns, and thereby increases the transient voltages which may occur between turns.

The waves entering this winding will traverse it and be reflected in it in much the same manner as in a transmission line. A reflection point corresponding to the closed end of a line will be found at the middle of the winding for single-phase or delta-connected transformers, with conjugate half waves arriving at opposite terminals at the same time, or at the further end for Y-connected transformers with grounded neutral. A reflection point corresponding to the open end of a line will be found at the middle of the winding for single-phase or delta-connected transformers with half waves of the same polarity arriving at opposite terminals at the same time.

With a wave train of given frequency entering the windings the position and character of the reflection point fixes the location of the nodes of the resulting standing wave train. Neglecting the effect of internal losses, this also determines the ratio of the resulting internal voltages to those existing in the line. This relation is illustrated in Figs. 2A, 2B and 2C, which represent standing waves of voltage in the winding in their relations

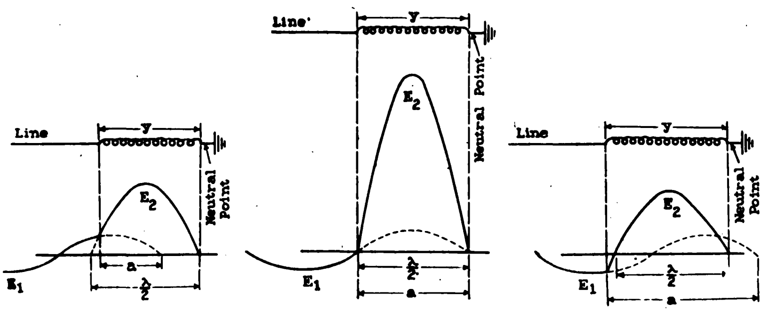


FIG. 2A FIG. 2B FIG. 2C

Standing waves of voltage in winding with uniformly distributed capacity, as related to standing waves in the line, for different values of $\frac{y}{\lambda} \cdot \frac{Z_2}{Z_1} = 10$.

to the standing waves in the line. Referring to these figures, we call the maximum value of voltage in the line E_1, and that in the winding E_2. The length of the winding from the entrance to the reflection point is y, and the wave length within the winding λ. Then, from the figures, we derive the equation

$$\frac{E_2}{E_1} = \frac{1}{\sqrt{\sin^2 2\,\pi \frac{y}{\lambda} + \left(\frac{Z_1}{Z_2}\right)^2 \cos^2 2\pi \frac{y}{\lambda}}} \qquad (1)$$

where Z_1, and Z_2 are the wave impedances respectively of the lines and the winding. Z_1 and Z_2 are further defined by the equation $Z_1 = \sqrt{\frac{L_1}{C_1}}$ and $Z_2 = \sqrt{\frac{L_2}{C_2}}$, where L_1 and C_1 are respectively the inductance and the capacity per unit length of the line, and L_2 and C_2 the effective inductance and the effective capacity per unit length of the winding. This equation is derived as follows:

The number of wave lengths between the reflection node and the entrance to the winding is $\frac{y}{\lambda}$, and a is the fraction of a wave length between the entrance and what would be the next node of same character as the reflection node, if the line were continuous. Let E and I be the voltage and current at the entrance, which are common to the line and the winding. Then, remembering that voltage nodes are current antinodes, and *vice versa*, we may write

$$E = E_2 \sin 2\pi \frac{y}{\lambda} = E_1 \sin a \qquad (2)$$

and

$$I = I_2 \cos 2\,\pi \frac{y}{\lambda} = I_1 \cos a \qquad (3)$$

or, since $$I_1 = \frac{E_1}{Z_1} \text{ and } I_2 = \frac{E_2}{Z_2}{}^* \qquad (4)$$

$$I = \frac{E_2}{Z_2} \cos 2\pi \frac{y}{\lambda} = \frac{E_1}{Z_1} \cos a$$

whence, by the relation $\cos = \sqrt{1 - \sin^2}$, we have

$$\sin^2 a = 1 - \left(\frac{E_2\,Z_1}{E_1\,Z_2} \cos 2\,\pi \frac{y}{\lambda}\right)^2 \qquad (5)$$

From (2) we have

$$\frac{E_2}{E_1} = \frac{\sin a}{\sin 2\,\pi \frac{y}{\lambda}} \qquad (6)$$

*See discussion on "Some Simple Examples of Transmission Line Surges," page 1641 of A.I.E.E. PROCEEDINGS, October, 1914.

Substituting sin a from (**5**) and simplifying, we get equation (**1**).

The way in which the internal voltages build up to the values shown in the figures and represented by equation (**1**) is explained as follows: With respect to the first wave of a traveling wave train, the transformer winding will act practically as open circuit to the line, regardless of the length of y, on account of the small amount of current admitted. That is, a voltage antinode and current node will be produced in the line at the entrance. Concentrating our attention upon figure 2B with $y = \frac{\lambda}{2}$, we find that the internal reflection will produce a voltage node at the entrance. The succeeding wave will therefore find as free admission as the first, and its current will be superposed upon that of the internal standing wave already produced. This process will continue, admitting more and more current from the line with each succeeding wave, until the current node found at first in the line at the entrance becomes an antinode, while the voltage antinode is reduced to a node.

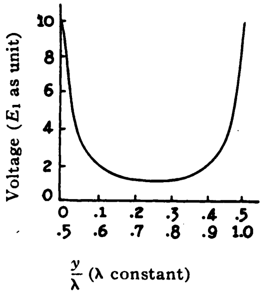


FIG. 3

Maximum voltage in winding with uniformly distributed capacity, as affected by the ratio of winding length to wave length, $\frac{y}{\lambda}$, in terms of the maximum standing wave voltage in the line. Plotted from equation (1), with $\frac{Z_1}{Z_2} = 10$.

For the particular case represented by figure 2B, which gives the maximum internal voltage, since $y = 1/2\ \lambda$, equation (**1**) simplifies to the form

$$\frac{E_2}{E_1} = \frac{Z_2}{Z_1} \text{ (for } y = 1/2\ \lambda) \tag{7}$$

The current in the winding is the same as the current in the line and the voltages are proportional to the respective wave impedances. Equation (**7**) holds for values of y such that $y = 1/2\ \lambda$ or any multiple thereof.

The maximum internal voltage which could be produced is plotted in Fig. 3 in terms of the external standing wave voltage E_1, for the ratio $\frac{Z_2}{Z_1} = 10$, with y varying between 0 and $\frac{1}{2}\lambda$. This curve repeats itself for the ranges of y between $\frac{1}{2}\lambda$ and λ.

between λ and $1\frac{1}{2}$ λ, etc. The minimum value for these voltages found for y equal $\frac{1}{4}$ λ or any odd multiple thereof, is

$$\frac{E_2}{E_1} = 1 \text{ (for } y = 1/4\ \lambda) \quad (8)$$

It is worthy of note that for values of $\frac{y}{\lambda}$ varying between 0 and $\frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$, etc., the current taken by the winding from the line lags 90 deg. in time behind the voltage at the transformer terminals, whereas for values between $\frac{1}{4}$ and $\frac{1}{2}$, $\frac{3}{4}$ and 1, etc., the current leads the voltage by 90 deg. That is, in the former cases the winding acts like an inductance, and in the latter like a capacity.

It should be noted, that, though the wave lengths as represented in the figures are the same in the winding as in the line, this is not true in ordinary linear measure. The linear relation between wave lengths is found as follows: The relation between frequency, velocity and wave length is

$$f = \frac{v}{\lambda} \quad (9)$$

where the velocity is

$$v = \frac{1}{\sqrt{L\,C}} \quad (10)$$

and L and C are the inductance and the capacity per unit length of the circuit. Since the frequency is the same for the line and the winding, using the subscripts 1 and 2 for the line and the winding respectively, we have

$$\lambda_2 = \frac{\sqrt{L_1\,C_1}}{\sqrt{L_2\,C_2}}\ \lambda_1 \quad (11)$$

The effect of frequency upon the internal voltages will become more obvious if we substitute in equation (1) the value of λ in terms of frequency, and inductance and capacity per unit length of the winding. This is

$$\lambda = \frac{1}{f\sqrt{L_2 C_2}} \quad (12)$$

The substitution gives

$$\frac{E_2}{E_1} = \frac{1}{\sqrt{\sin^2 2\,\pi f y \sqrt{L_2 C_2} + \left(\frac{Z_1}{Z_2}\right)^2 \cos^2 2\,\pi f y \sqrt{L_2 C_2}}} \quad (\mathbf{13})$$

We must still consider, moreover, the effects of frequency upon L_2 and C_2, since, at high frequencies, the effective inductance and the effective capacity per unit length of the winding are affected by the frequency.

The turn to turn capacity between positive and negative half waves becomes important at high frequencies, on account of the proximity of the half waves. This capacity has the same effect upon velocity, wave length, wave impedance and voltage as twice the same amount of capacity to ground, and must therefore be added to the capacity to ground at double value.

The increase in total effective capacity per unit length with increased frequency, due to this cause, is all the greater on account of its reflex action. By reducing the velocity of propagation, it brings the half waves still closer together, with consequent further increase in effective capacity.

On the other hand, the effective inductance per unit length of the winding will be reduced by increased frequency. This is due to the fact that the part of the winding which acts as a unit with respect to inductance is reduced by the shorter wave length. This effect is illustrated by Nagaoka's table of correction factors for inductance calculated for a single layer coil by the formula for a long solenoid. This table, given in " Calculation of Alternating Current Problems " by Cohen, pages 80 and 81, gives correction factors varying from unity for the infinitely long solenoid to 0.2 for a coil of length 0.1 as great as its diameter.

This reduction in inductance with increased frequency will tend to neutralize the effect of increased capacity upon the velocity of propagation, but augments its effect upon the wave impedance of the winding. If we assume that the factor by which the capacity is increased is the same as the factor by which the inductance is decreased, the sine and cosine terms in equation (**13**) will not be affected, the only effect upon the internal voltages due to the variations in capacity and inductance with frequency being that which appears in the factor $\left(\frac{Z_1}{Z_2}\right)^2$, which may be written $Z_1^2 \frac{C_2}{L_2}$.

Where L_w and C_w are the total inductance and capacity of the winding measured at normal operating frequency, if we assume that the wave impedance of the winding is $\sqrt{\frac{L_w}{C_w}}$ for the frequency $f = \frac{1}{\sqrt{L_w C_w}}$, which gives a ½ wave length within the winding, and if we assume further that, due to the changes in inductance and capacity, the wave impedance is inversely proportional to the square root of the frequency,* we may write, for any frequency,

$$Z_2^2 = \frac{L_2}{C_2} = \frac{1}{2 f \sqrt{L_w C_w}} \frac{L_w}{C_w} \qquad \textbf{(14)}$$

We have also, when y is the total length of the winding

$$y \sqrt{L_2 C_2} = \sqrt{L_w C_w} \qquad \textbf{(15a)}$$

and when y is ½ of the length of the winding

$$y \sqrt{L_2 C_2} = \frac{\sqrt{L_w C_w}}{2} \qquad \textbf{(15b)}$$

Substituting **(14)** and **(15a)** in **(13)** we obtain

$$\frac{E_2}{E_1} = \frac{1}{\sqrt{\sin^2 2\pi f \sqrt{L_w C_w} + \frac{2 Z_1^2 f C_w^{3/2}}{L_w^{1/2}} \cos^2 2\pi f \sqrt{L_w C_w}}} \qquad \textbf{(16a)}$$

or from **(14)** and **(15b)**

$$\frac{E_2}{E_1} = \frac{1}{\sqrt{\sin^2 \pi f \sqrt{L_w C_w} + \frac{2 Z_1^2 f C_w^{3/2}}{L_w^{1/2}} \cos^2 \pi f \sqrt{L_w C_w}}} \qquad \textbf{(16b)}$$

The internal voltages in terms of the line voltage, in accord-

*This is equivalent to the assumption that L_2 is inversely and C_2 directly proportional to the square root of the frequency.

ance with equations (**16**), are plotted in Fig. 4, for the assumed values of

$$Z_1 = 490 \text{ ohms}$$
$$C_w = 0.00248 \text{ mf.}$$
$$L_w = 0.175 \text{ henrys}$$

The effects of losses within the transformer winding were not taken into account in the above derivations. The dielectric losses at high frequency and high voltage are high, and cause a rapid damping of the entering wave train as it traverses the winding. The outgoing wave being smaller than the incoming one, this gives a combination of traveling waves and standing waves,

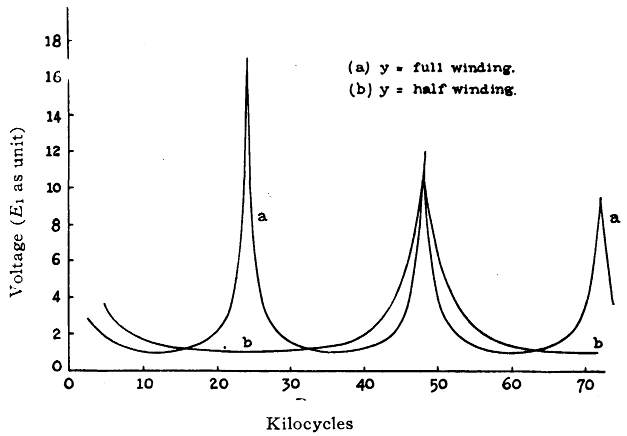


FIG. 4

Effect of frequency upon the maximum voltage in winding with uniformly distributed capacity, in terms of the maximum standing wave voltage in the line. Plotted from equation (**16b**), with $Z_1 = 490$ ohms, $L_w = 0.175$ henry and C_w =0.00248 m.f.

the standing waves being smaller the greater the damping. This not only prevents the internal voltages from building so high, but throws them out of phase with each other. With a true standing wave train, all voltages are in the same time phase, and either add or subtract numerically, but voltages measured between points equally distant in the winding have every value from zero to maximum. With a pure undamped traveling wave, the same voltage may be measured between points equally distant in the winding, but all phase relations are found. With traveling waves superposed upon standing waves, we find both varying voltages and varying phase relations between equidistant points. This is what we will expect to find in a transformer winding of this type, the voltages being most nearly in phase near the node of reflection, where the standing waves predominate and furthest out of phase, but almost nearly equal, between equidistant points near the entrance, where the traveling wave component is maximum.

Winding With Capacity More or Less Concentrated and Localized. A winding consisting of groups of pancake coils interlaced with a low-voltage winding is represented by the simplified arrangement of capacity and inductance in Fig. 5. If the middle of this winding is grounded, or a neutral point, the middle capacity C_3 is short circuited. Neglecting the capacities C_1 and C_5, adjacent to the line, which exist also with the winding with distributed capacity considered above, and also neglecting the turn-to-turn and coil-to-coil capacities, which are in parallel with the respective inductances, we will consider a disturbance

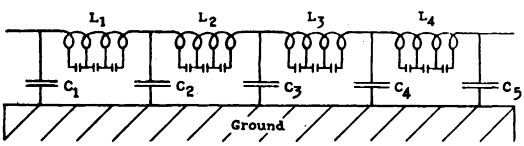
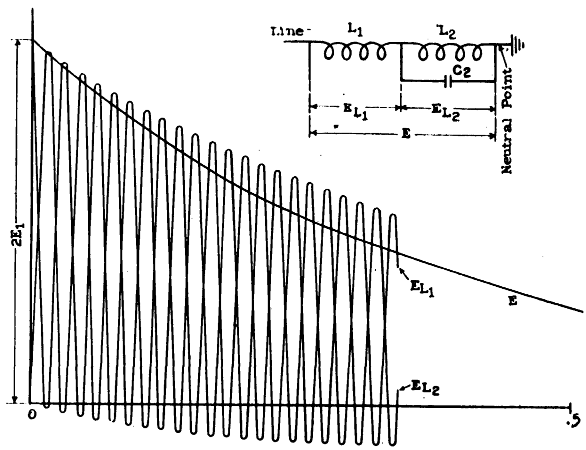


FIG. 5
Diagram of capacity and inductance in winding with localized capacity.

entering this winding from one end. We find between the line and neutral, or ground, the inductance L_1 in series with the parallel arrangement of inductance L_2 and capacity C_2. Any other location of neutral point or ground, as with one end of the winding grounded, will give a more complicated arrangement of inductance and capacity between this point and the line. The behavior of such combinations is, however, based upon the same principles as those which we will consider in detail in connection with this most simple arrangement.

An impulse or wave of sufficiently short duration impinging upon this combination would be practically unfelt beyond the inductance L_1, since the current in the inductance would be zero at the first instant, and current must flow to charge the capacity C_2. If the inductances and capacities were distinct concentrated quantities, as assumed, the voltage of this wave would be uniformly distributed between the turns of inductances L_1. There is, however, a certain amount of capacity to ground distributed with the inductance. In fact, a large part of the capacities shown concentrated in the figure are distributed near the ends of the inductances. This would effect the concentration of the voltage of an abrupt wave front over a very small number of turns, and in the ultimate conceivable limit, with a perfectly sheer wave front, over a single turn, if it were not for the fact that we have also capacities between turns. As in the winding already considered, these capacities between turns always effect the distribution of an abrupt voltage over a considerable number of turns, the voltage between adjacent turns being smaller, the larger the capacity between turns.



Time in milliseconds

FIG. 6

Voltage oscillations set up by a traveling wave with abrupt front, in a winding with capacity localized as shown in the sketch. Plotted from equations (**36a**) and (**37a**), with $Z_1 = 490$ ohms, $L_1 = L_2 = 0.1$ henry and $C_2 = 0.0005$ m.f.

With a wave of considerable duration, an oscillation is produced, which is investigated mathematically in an appendix to this paper, the resulting voltages across L_1 and L_2 being plotted in Fig. 6 for the values $L_1 = L_2 = 0.1$ henry, $C_2 = 0.0005$ mf. and $Z_1 = 490$ ohms.

If a wave of opposite polarity from the first one appears at the terminals of the winding at the end of the first or any odd numbered half cycle of the oscillation due to the first wave front shown in Fig. 6, a new oscillation will be superposed upon the first one in phase with it, with a corresponding increase in amplitude. With no damping in the winding, the amplitude of the voltages which might be built up in this manner by a succession of wave fronts of opposite polarity, so timed as to be in resonance with the oscillations of the winding, depends upon the current supplied by the line. The reflection of the first wave at the transformer terminal produces a voltage antinode and current node at the entrance of the winding. As the resonant voltages build up across the parts of the winding, the high frequency current taken by the winding increases, gradually changing the voltage antinode and current node at the entrance to a voltage node and current antinode. This supplies the maximum current which can be delivered by the line and consequently limits the resonant voltages produced.

The frequency of the oscillations expressed in equations (**36**) and (**37**) of the appendix, and represented in Fig. 6, at which resonant voltages may be built up as described above, is the same as the frequency of resonance which would be calculated from the ordinary impedance equations. Thus, for the parallel combination of L_2 and C_2, at frequencies above its resonant frequency, the equivalent capacity impedance is (writing w for $2 \pi f$).

$$\frac{1}{w\,C'} = \frac{1}{w\,C_2 - \dfrac{1}{w\,L^2}} = \frac{w\,L_2}{w^2\,L_2\,C_2 - 1} \qquad \textbf{(17)}$$

This equivalent capacity resonates with the series inductance L_1 at a frequency giving

$$w L_1 = \frac{1}{w C'} \qquad \textbf{(18)}$$

and from equations (**17**) and (**18**) we obtain

$$w = \sqrt{\frac{L_1 + L_2}{L_1\,L_2\,C_2}} \qquad \textbf{(19)}$$

Or, when $L_1 = L_2$ dropping subscripts

$$w = \sqrt{\frac{2}{LC}} \qquad \textbf{(20)}$$

This is the value of c in equations (**36b**) and (**37b**) as shown in (**36a**) and (**37a**) of the appendix, and the frequency found in the oscillations of Fig. 6 is

$$f = \frac{1}{2\pi}\sqrt{\frac{2}{LC}} \tag{21}$$

Assuming that L_1, L_2 and C_2 are not affected by frequency, and neglecting the internal losses of the winding, we may calculate the maximum voltages which would be produced by a sustained wave train of any frequency. As with the winding with uniformly distributed capacity, we will call maximum standing wave voltage and current in the line E_1 and I_1 and voltage and current at the entrance of the winding E and I. As in equations (**2**) and (**3**)

$$E = E_1 \sin a \tag{22}$$

and

$$I = I_1 \cos a \tag{23}$$

"a" being the fraction of a wave length in the line from the entrance of the winding to what would be a nodal point corresponding to a short circuited line. In the winding the voltage E must force the current I through the impedance

$wL_1 + \dfrac{wL_2}{1 - w^2 L_2 C_2}$, so that

$$E_1 = I\left(wL_1 + \frac{wL_2}{1 - w^2 L_2 C_2}\right) \tag{24}$$

Substituting (**22**) and (**23**) this gives

$$E_1 \sin a = I_1 \cos a \left(wL_1 + \frac{wL_2}{1 - w^2 C_2 L_2}\right) \tag{25}$$

whence

$$\tan a = \frac{1}{Z_1}\left(wL_1 + \frac{wL_2}{1 - w^2 L_2 C_2}\right) \tag{26}$$

Having found a, the voltages across L_1 and L_2 are

$$E_{L_1} = I_1 \cos a \; w L_1 \text{ (max. value)} \tag{27}$$

and

$$E_{L_2} = I_1 \cos a \frac{w L_2}{1 - w^2 L_2 C_2} \text{ (max. value)} \tag{28}$$

Or, in terms of the maximum voltage in the line,

$$\frac{E_{L_1}}{E_1} = \frac{w L_1}{Z_1} \cos a \tag{29}$$

and

$$\frac{E_{L_2}}{E_1} = \frac{w L_2}{Z_1 (1 - w^2 L_2 C_2)} \cos a \tag{30}$$

The gradient of maximum instantaneous voltage within the winding, and the standing wave voltage in the line, in accordance with equations (**26**), (**29**) and (**30**), are shown in Fig. 7 for the values

$$Z_1 = 490 \text{ ohms}$$
$$L_1 = L_2 = 0.1 \text{ henry}$$
$$C_2 = 0.0005 \text{ mf.}$$

and for various values of frequency selected with view to illustrating its effect upon internal voltages. L_1, L_2 and C_2 are assumed to be constant independent values (not affected by frequency). A feature of incidental interest appears in the location of the voltage node (current antinode) in the line with respect to the entrance to the winding.

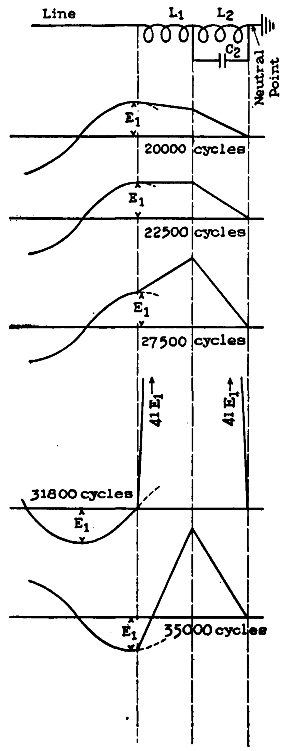


FIG. 7
Voltages in winding with capacity localized as shown, as related to the standing waves in the line, for different frequencies. Calculated from equations (**29**) and (**30**), with $Z_1 = 490$ ohms, $L_1 = L_2 = 0.1$ henry and $C_2 = 0.0005$ mf.

We find the voltage antinode exactly at the entrance to the winding at the frequency at which L_2 and C_2 are in resonance with each other. This, for the value of inductance and capacity used, is

$$f = \frac{1}{2 \pi \sqrt{L_2 C_2}} = 22{,}500 \text{ cycles.}$$

Below this frequency this parallel combination acts like an inductance, of value increasing as the frequency increases and reaching infinity

$$\frac{wL_2}{1 - w^2{}_{L^2\,C_2}} = \frac{wL_2}{0} = \infty \qquad (31)$$

at this resonant frequency. Above this frequency it acts like a capacity, of value increasing from zero at resonant frequency to the limiting value C_2 at very high frequencies.

Below the frequency $f = \frac{1}{2\,\pi\,\sqrt{L_2\,C_2}}$ the total arrangement L_1, L_2 and C_2 acts as an inductance, the inductive impedance $\frac{wL_2}{1 - w^2\,L_2\,C_2}$ being in series with $w\,L_1$. The voltage E_{L_1} and E_{L_2} are in phase with each other, the value of the former diminishing to zero at this frequency, where the infinite impedance $\frac{wL_2}{1 - w^2\,L_2\,C_2} = \infty$ acts like open circuit. Between this frequency and the resonant frequency $f = \frac{1}{2\,\pi}\sqrt{\frac{L_1 + L_2}{L_1\,L_2\,C_2}}$ at which the voltage node appears at the terminals, the total arrangement acts like a condenser, with value increasing from zero at the lower frequency to infinity at the higher frequency. The voltages E_{L_1} and E_{L_2} are in phase opposition, the voltage at the winding terminals being equal to their difference. The voltage E_{L_1} grows from zero value at the lower frequency to a value limited by the maximum value of current in the line at the higher frequency. The voltage E_{L_2}, which is the capacity or leading voltage, is greater than the voltage E_{L_1} by the amount of the voltage in the line at the winding terminals. Above the frequency $f = \frac{1}{2\,\pi}\sqrt{\frac{L_1 + L_2}{L_1\,L_2\,C_2}}$ the voltages E_{L_1} are E_{L^2} and still in phase opposition, but the voltage E_{L_1} is larger than E_{L_2} by the amount of the voltage at the winding terminals, and the total arrangement is acting like an inductance, increasing from zero to the limiting value L_1. At very high frequencies the voltage E_{L_2} becomes very small, the voltage E_{L_1} being practically equal to the voltage in the line.

The variations in these internal voltages with the frequency, and that of the voltage across the total combination, are shown in the curves of Fig. 8.

We have discussed in detail the behavior of the arrangement L_1, L_2 and C_2. If the same winding (Fig. 5) were grounded at one end, or at some other point not the middle, or if a winding of more than four groups (six for instance) were grounded at the middle, assuming a simplification of the distribution of capacity such as shown in Fig. 5, there would still be a more complicated arrangement than the one we have considered. As already stated however, the same general principles would be involved in the behavior of any such arrangement. Detailed discussion is not

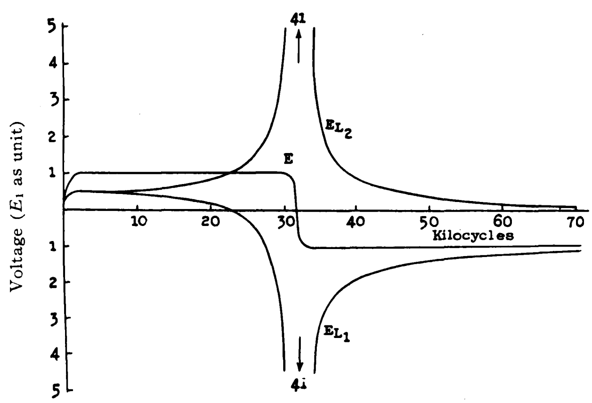


FIG. 8

Effect of frequency upon voltages in winding with capacity localized as shown in Figure 7, in terms of the maximum standing wave voltage in the line. Plotted from equations (**29**) and (**30**), with Z_1 = 490 ohms, $L_1 = L_2 = 0.1$ henry, and C_2 = 0.0005 mf.

given therefore, to any other such arrangement, but, by way of further illustration of what may occur, we give in Fig. 9 a set of diagrams similar to those of Fig. 7, showing the relation of internal voltages to the line voltage at various selected frequencies, with three groups of coils (three units of inductance, with intervening capacities) between the line and a grounded or neutral point. The equations for this case, to which these diagrams correspond, are,

$$\tan a = \frac{1}{Z_1}\left[wL_1 + \frac{w(L_2 + L_3) - w^3 L_2 L_3 C_3}{1 - w^2 (L_2C_2 + L_3C_3 + L_3C_2) + w^4 L_2 L_3 C_2 C_3}\right] \quad \textbf{(32)}$$

$$\frac{E_{L_1}}{E_1} = \frac{1}{Z_1} w L_1 \cos a \tag{33}$$

$$\frac{E_{L_2}}{E_1} = \frac{1}{Z_1} \frac{w L_2 - w^3 L_2 L_3 C_3}{1 - w^2 (L_2 C_2 + L_3 C_3 + L_3 C_2) + w^4 L_2 L_3 C_2 C_3} \cos a \tag{34}$$

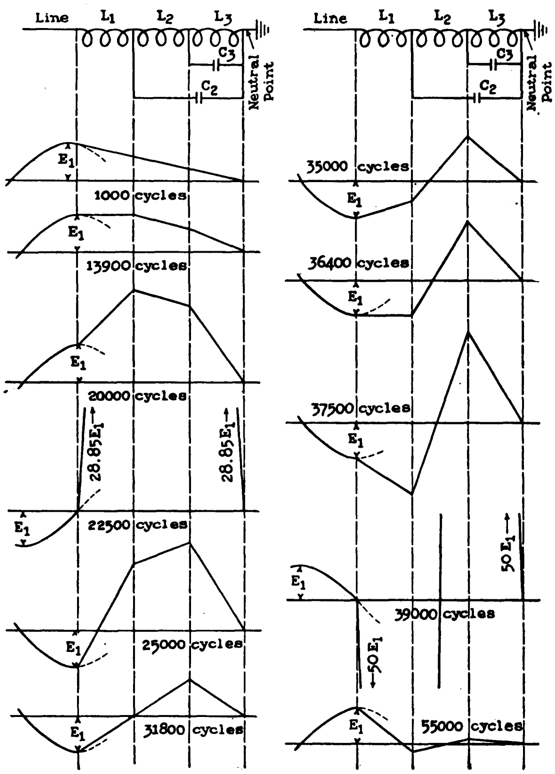


FIG. 9

Voltages in winding with capacity localized as shown, as related to standing waves in the line, for different frequencies. Calculated from equations (**37**), (**38**) and (**39**), with Z_1 = 490 ohms, $L_1 = L_2 = L_3 = 0.1$ henry and $C_2 = C_3 = 0.0005$ mf.

and

$$\frac{E_{L_3}}{E_1} = \frac{1}{Z_1} \frac{w L_3}{1 - w^2 \,(L_2 C_2 + L_3 C_3 + L_3 C_2) + w^4 L_2 L_3 C_2 C_3} \cos a \tag{35}$$

With all inductance units equal and all capacity units equal we drop the subscripts and write:

$$\tan a = \frac{1}{Z_1} \frac{3wL - 4w^3 L^2 C + w^5 L^3 C^2}{1 - 3w^2 LC + w^4 L^2 C^2} \tag{36}$$

$$\frac{E_{L_1}}{E_1} = \frac{1}{Z_1} w L \cos a \tag{37}$$

$$\frac{E_{L_2}}{E_1} = \frac{1}{Z_1} \frac{wL - w^3 L^2 C}{1 - 3w^2 LC + w^4 L^2 C^2} \cos a \tag{38}$$

and

$$\frac{E_{L_3}}{E_1} = \frac{1}{Z_1} \frac{wL}{1 - 3w^2 LC + w^4 L^2 C^2} \cos a \tag{39}$$

Several cases of resonance, at different frequencies, are found in these diagrams. It will be noticed that high voltages occur across some of the inductance units, at different frequencies, but not always across all at the same frequency. The variations in the internal voltages of this combination with the frequency are shown in Fig. 10.

As with the winding of distributed capacity, the damping due to the internal losses prevents the building up of the excessive voltages found above. These voltages can be built up only by the admission of small amounts of energy by the inductance L_1 from the successive wave fronts of a high-frequency train. The dielectric losses in the winding increase as the voltage builds up, until the energy absorbed is equal to the energy admitted. These losses are even higher in windings of this type than in those with distributed inductance and capacity, and probably restrict the voltages to a small fraction of those found above.

It has already been mentioned that in any case a certain amount of capacity will be found distributed with the inductances L_1, L_2, etc. It is obvious, therefore, that at frequencies sufficiently high, these parts of this winding may behave in a manner somewhat similar to the winding with distributed inductance and capacity. That is, standing waves may be set up within the individual coils or groups of the winding. Frequencies giving these results are produced by discharges at or near the terminals of the transformer, and such discharges produce the most dangerous condition with respect to the insulation between turns. On the other hand, the most dangerous condition with respect to the insulation between the winding and ground (low-voltage winding and core) is produced by the frequencies producing resonance between groups. The frequencies producing dangerous voltages between turns are much higher than these.

Effects of Normal Frequency Currents and Voltages. No consideration has been given in the foregoing discussions to the effects of the normal voltages and currents existing in the windings of the transformer before the arrival of the steep wave front and the high frequency wave train. A statement of the principle

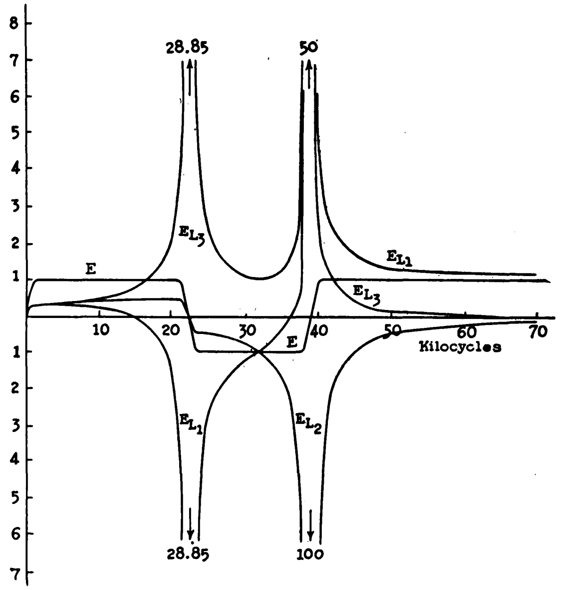


FIG. 10

Effect of frequency upon voltages in winding with capacity localized as shown in Figure 9, in terms of maximum standing wave voltage in the line. Plotted from equations (**37**), (**38**) and (**39**), with $Z_1 = 490$ ohms, $L_1 = L_2 = L_3 =$ 0.1 henry and $C_1 = C_2 = 0.005$ mf.

facts involved will be sufficient answer for the questions arising in this direction.

It is evident that the distribution of voltage in a circuit or winding can not be disturbed or altered by the flowing of currents of equal value throughout the entire circuit, or by any changes in current which occur uniformly throughout the entire circuit, so that at any given instant the same value of current flows at every point in the circuit. Even the normal variations of voltage within a transformer winding having capacity are accompanied by the flow of charging currents which traverse only portions of the winding to supply the charges corresponding to the voltage changes of the capacity which is located within the winding. These currents are normal frequency currents, controlled by the normal flux variations which bind the charges supplied. These currents are ordinarily very small as compared with the normal frequency load or exciting currents which flow with equal value throughout the winding and upon which they are superposed.

Now the current of a traveling wave entering the winding is only that required to change the potential of the winding to correspond with the voltage of the wave. The magnitude of this current will not be affected by currents already flowing with equal value throughout the winding. Moreover, the normal frequency charging current of the winding is negligibly small as compared with that of the traveling wave and need not be considered. The wave current is merely superposed upon the previously existing current.

The wave current is still small as compared with the normal load current of the transformer, so that if a transformer is loaded, the entrance of the first wave of an oscillation has small effect upon the actual resultant current. High frequency currents of considerable value might, as we have seen, be built up within the winding by sustained wave trains or oscillations at resonating frequencies, except for the heavy internal damping due to dielectric loss at these high frequencies and with the high accompanying voltages. Due to this damping, these currents are still relatively small. In any event they may still be looked upon as merely superposed upon the current with equal value throughout the winding, produced by the voltage impressed from normal source, although this latter current may become changed from its initial value. It does, in fact, change in normal operation, since it is an alternating current, but it may also be changed by the conditions producing the oscillation, as by short circuit, for instance. Thus, in the case of short circuit beginning with an oscillation, we have the oscillatory current superposed upon the current from the generator which has uniform value throughout the transformer winding at any instant, but which changes cyclically in time with the normal frequency of the generator, and more or less gradually from the initial to the final value.

Questions as to the effects of the initial or normally changing voltages or voltage distributions of the windings upon the voltages set up by waves and oscillations have already been answered implicitly in the above discussion of the effects of currents, on account of the perfectly definite relation between voltages and currents in any circuit with respect either to steady or gradually changing conditions or to oscillations. The statement may be made general, therefore, with respect to both voltage and current, that the initial current and voltage of the transformer will have no effects upon the results of a steep wave front or high-frequency wave train except in so far as they may fix the conditions which set up the oscillation, and so determine its character, and in so far as the actual conditions of current and voltage set up are due to the superposition of the oscillatory or traveling wave currents and voltages upon a value of current and a voltage gradient which is uniform throughout the winding and which is arrived at by a process of gradual change from the initial value.

Effect of the Core and its State of Magnetization. It is necessary to consider the effect of the transformer core and its state of magnetization upon the behavior of the windings when subjected to a high-frequency disturbance. The statement of some fundamental facts will help to clear up these questions also.

The variation of the flux in the core in response to the voltage applied at normal operating frequencies, and the relation of the magnetic density to the exciting current, are well understood. It is also known that the core responds in the same general manner at high frequencies, the chief differences between the behavior at high frequency and at low frequency being the apparent reduction in permeability due to the restriction of flux from the center of the sheets by skin effect, and the increased eddy current and hysteresis losses for a given flux variation. None of these considerations are important, however, since the flux variations corresponding to any high-frequency voltage which may be impressed will always be small.

The behavior of the windings with respect to the core at high frequency will differ from that at operating frequencies in that a small part of one of the windings may act as primary with respect not only to the other winding but also with respect to the remaining part of the same winding. Moreover the portion of the winding acting as primary is variable from instant to instant. Thus if we consider a traveling wave entering the winding with distributed capacity, the first turns act as primary with respect to all the other turns, whereas a little later the number of turns acting as primary turns will have increased. As the entering wave traverses the first part of the winding, the distributed capacity permits the setting up of reverse currents in the remaining part of the winding, as well as in the opposite winding, which correspond to a condition of short circuited secondary, even though the other windings were open circuited. The result is that the flux set up by the wave is practically all leakage flux occupying only a small portion of the core. If the wave be a long one, the secondary currents soon cease, with the charging of the available capacity, and the condition is changed from one of short circuit to one of open circuit. The flux due to the traveling wave thus comes to occupy the complete magnetic circuit of the core, resulting in a large increase in the inductance per unit length of winding. If the other winding be connected to a closed circuit, these conditions will be affected only in so far as current can be drawn from the other circuit. This requires voltage, and involves the transformation of the wave from one winding to the other.

If we consider the case of standing waves set up within the winding, the amplitude of all of the waves being the same, we will in general find a fractional excess of positive or negative half waves of current within the winding.

This gives an excess of positive or negative ampere turns which magnetizes the core, and so generates a voltage throughout the entire winding which is counter to the voltage in the line at the transformer terminals. The internal standing wave voltages are superposed upon this voltage. Within certain ranges of frequency or lengths of winding, *i.e.*, of the ratio $\frac{y}{\lambda}$, the standing wave current entering the winding is restricted and consequently the internal standing wave voltages restricted, by the condition that the distributed voltage can not exceed the standing wave voltage found in the line at the terminals of the transformer. For the frequency giving the maximum internal voltages, however, *i.e.*, with a voltage node and current antinode at the entrance, there is no such restriction, since there are equal numbers of positive and negative half waves of current within the winding.

Similar restrictions will be found with the winding in which capacity and inductance are separated into alternate more or less concentrated amounts, but will not appear at resonant frequencies, since equal amounts of positive and negative current will be found within the inductances of the winding.

The high frequency flux set up within the core by an excess of positive or negative ampere turns generates voltages not only in the winding upon which the high frequency disturbance is impressed but also in the other winding. Charging current set up by this voltage in the second winding is in phase with the excitation, and tends to increase the voltage. Excess voltages may be thus built up in this winding at its resonant frequency.

While the transformer core no doubt has an important influence on the behavior of the windings at high frequency, this influence is not affected by its initial state of magnetization except in so far as its permeability is affected. The voltages generated by flux in the core are distributed throughout the windings, and depend in a regular manner upon the unbalanced ampere turns and the high-frequency permeability of the core, in its existing state of saturation.

Conclusion

The above analysis of the behavior of transformer windings is by no means complete, but such an examination of the fundamental facts and principles involved gives us a clearer insight into the nature of the excessive internal voltages which, as shown by experience, are produced in practise. This will enable us to guard against these voltages in a more scientific and economical manner. It is expected that this will constitute the subject of a subsequent paper.

APPENDIX

The behavior of the combination of inductance and capacity represented in Fig. 6, at the end of a transmission line, when a wave with steep front and of considerable length strikes it, is investigated as follows:

The reflection in the line is at the first instant complete, as at the open end of a line, giving double voltage and zero current. If the voltage of the original wave is E_1 and the current I_1, the reflected voltage and current at the first instant are $(E_1')_0 = E_1$ and $(I_1')_0 = -I_1$. At a subsequent instant the numerical value of the reflected current is reduced by the current flowing through the inductance L_1, and the voltage impressed on the transformer is correspondingly reduced.

The voltage at the terminals of the transformer to any instant is

$$E = E_1 + E_1' \tag{1}$$

This voltage appears across the inductance L_1 and the parallel arrangement L_2 C_2, so that we also have

$$E = L_1 \frac{dI_{L_1}}{dt} + L_2 \frac{dI_{L_2}}{dt} \tag{2}$$

where I_{L_1} and I_{L_2} are the respective currents flowing in inductances L_1 and L_2.

In the line we have

$$E_1 = I_1 Z_1 \tag{3}$$

and

$$E_1' = - I_1' Z_1 \tag{4}$$

and the current in the inductance L_1 is

$$I_{L_1} = I_1 + I_1' \tag{5}$$

so that

$$- I_1' = I_1 - I_{L_1}$$

This value in (**4**) gives

$$E_1' = (I_1 - I_{L_1}) Z_1 \tag{6}$$

whence

$$E_1' = E_1 - I_{L_1} Z_1 \tag{7}$$

Substituting this value in (**1**) gives

$$E = 2 E_1 - I_{L_1} Z_1 \tag{8}$$

and this with equation (**2**) gives

$$2E_1 = L_1 \frac{dI_{L_1}}{dt} + L_2 \frac{dI_{L_2}}{dt} + I_{L_1} Z_1 \tag{9}$$

Now, considering voltage in the parallel elements C_2 and L_2, we have

$$\int \frac{I_{C_2}\, dt}{C_2} = L_2 \frac{dI_{L_2}}{dt} \tag{10}$$

whence

$$I_{C_2} = C_2 L_2 \frac{d^2 I_{L_2}}{dt^2} \tag{11}$$

We also have

$$I_{C_2} + I_{L_2} = I_{L_1} \tag{12}$$

whence

$$I_{L_1} = I_{L_2} + C_2 L_2 \frac{d^2 I_{L_2}}{dt^2} \tag{13}$$

Substituting (**13**) in (**9**), and transforming.

$$\frac{d^3 I_{L_2}}{dt^3} + \frac{Z_1}{L_1} \frac{d^2 I_{L_2}}{dt^2} + \frac{L_1 + L_2}{L_1 L_2 C_2} \frac{d I_{L_2}}{dt} + \frac{Z_1}{L_1 L_2 C_2} I_{L_2} = \frac{2 E_1}{L_1 L_2 C_2} \tag{14}$$

and substituting

$$I = I_{L_2} - \frac{2E_1}{Z_1} \tag{15}$$

we get

$$\frac{d^3 I}{dt^3} + \frac{Z_1}{L_1} \frac{d^2 I}{dt^2} + \frac{L_1 + L_2}{L_1 L_2 C_2} \frac{dI}{dt} + \frac{Z_1}{L_1 L_2 C_2} I = 0 \tag{16}$$

The solution of this equation is

$$I = A_1 e^{x_1 t} + A_2 e^{x_2 t} + A_3 e^{x_3 t} \tag{17}$$

x_1, x_2 and x_3 being the roots cf the auiary equation.

$$x^3 + \frac{Z_1}{L_1} x^2 + \frac{L_1 + L_2}{L_1 L_2 C_2} x + \frac{Z_1}{L_1 L_2 C_2} = 0 \tag{18}$$

It is known that all of these roots are negative, since all of the co-efficients of equation (**18**) are positive. We may, however, substitute positive numerical values with the negative sign prefixed. Instead of equation (**17**), then, we may write

$$I = A_1 e^{-a_1 t} + A_2 e^{-a_2 t} + A_3 e^{-a_3 t} \tag{19}$$

in which a_1, a_2 and a_3 will be positive.

For the ranges of values of the constants appearing in the coefficients of equation (**18**), it will contain a pair of complex imaginary roots. In this case, instead of a_1, a_2 and a_3, we may write a, $(b - jc)$ and $(b + jc)$, and for purposes of calculation, it will be convenient to write equation (**19**) in trigonometric form. Thus, instead of

$$I = A_1 e^{-at} + A_2 e^{-(b-jc)t} + A_3 e^{-(b+jc)t} \tag{20}$$

we have

$$I = A e^{-at} + e^{-bt} (B \cos ct + C \sin ct) \tag{21}$$

This equation represents a condition of damped oscillation superposed upon a condition of decay.

It is not easy to obtain the roots of equation (**18**) in terms of the constants involved in the equation, but the numerical values of the constants may be substituted for any particular case, and the solution obtained for the particular case. It is necessary, also, to determine the constants A_1, A_2 and A_3 or A, B and C. Having determined the constants A_1, A_2 and A_3; A, B and C may be determined independently, or from the relations

$$\left.\begin{aligned} A &= A_1 \\ B &= (A_2 + A_3) \\ \text{and} \quad C &= j\,(A_2 - A_3) \end{aligned}\right\} \tag{22}$$

For determining these constants, since I_{L_2} and I_{C_1} are zero at the first instant, equations (**15**) and (**11**) give

$$\text{For } t = 0 \quad \left\{ \begin{aligned} I &= -\frac{2E_1}{Z_1} \\ \frac{dI}{dt} &= 0 \\ \text{and } \frac{d^2 I}{dt^2} &= 0 \end{aligned} \right. \tag{23}$$

Substituting these values in (**19**) and its derivatives, we obtain

$$A_1 + A_2 + A_3 = -\frac{2\,E_1}{Z_1} \tag{24}$$

$$a_1 A_1 + a_2 A_2 + a_3 A_3 = 0 \tag{25}$$

and

$$a_1^2 A_1 + a_2^2 A_2 + a_3^2 A_3 = 0 \tag{26}$$

whence

$$A_1 = \frac{a_2\,a_3}{(a_3 - a_1)\,(a_1 - a_2)}\,\frac{2\,E_1}{Z_1} \tag{27}$$

$$A_2 = \frac{a_1\,a_3}{(a_1 - a_2)\,(a_2 - a_3)}\,\frac{2\,E_1}{Z_1} \tag{28}$$

and

$$A_3 = \frac{a_1\,a_2}{(a_2 - a_3)\,(a_3 - a_1)}\,\frac{2\,E_1}{Z_1} \tag{29}$$

Substituting these values in equation (**19**) we have

$$I = \frac{2E_1}{Z_1}\left[\frac{a_2\,a_3}{(a_3 - a_1)\,(a_1 - a_2)}\,e^{-a_1 t} + \frac{a_1\,a_3}{(a_1 - a_2)\,(a_2 - a_3)}\,e^{-a_2 t} + \frac{a_1\,a_2}{(a_1 - a_3)\,(a_3 - a_1)}\,e^{-a_3 t}\right] \tag{30}$$

Now we wish to determine the voltages across the inductance L_1 and L_2. These are

$$E_{L_1} = L_1\,\frac{dI_{L_1}}{dt} \text{ and } E_{L_2} = L_2\,\frac{dI_{L_2}}{dt} \tag{31}$$

For the latter, we obtain from equation (**15**)

$$\frac{dI_{L_2}}{dt} = \frac{dI}{dt} \tag{32}$$

whence, substituting the value of I from (**30**), we have

$$E_{L_2} = -2E_1 \frac{L_2}{Z_1}\left[\frac{a_1 a_2 a_3}{(a_3 - a_1)(a_1 - a_2)} e^{-a_1 t} + \frac{a_1 a_2 a_3}{(a_1 - a_2)(a_2 - a_3)} e^{-a_2 t} + \frac{a_1 a_2 a_3}{(a_2 - a_3)(a_3 - a_1)} e^{-a_3 t}\right] \quad (33)$$

We now obtain $L_1 \frac{dI_{L_1}}{dt}$ from equation (**13**).

Thus

$$L_1 \frac{dI_{L_1}}{dt} = L_1 \frac{d}{dt}\left(I_{L_2} + C_2 L_2 \frac{d^2 I_{L_2}}{dt^2}\right) \quad (34)$$

Substituting the quantities in the parenthesis from equations (**15**) and (**30**), performing the required differentiations and simplifying, we have

$$E_{L_1} = -2E_1 \frac{L_1}{Z_1}\left[\frac{(1 + C_2 L_2 a_1^2)\, a_1 a_2 a_3}{(a_3 - a_1)(a_1 - a_2)} e^{-a_1 t} + \frac{(1 + C_2 L_2 a_2^2)\, a_1 a_2 a_3}{(a_1 - a_2)(a_2 - a_3)} e^{-a_2 t} + \frac{(1 + C_2 L_2 a_3^2)\, a_1 a_2 a_3}{(a_2 - a_3)(a_3 - a_1)} e^{-a_3 t}\right] \quad (35)$$

When the values of the constants in equation (**18**) are such as to give two complex imaginary roots, equations (**33**) and (**35**) are both transformed to the trigonometric form. The proper constants may be obtained directly from those of equations (**33**) and (**35**) by the relations given in (**22**). The roots a_2 and a_3 now being of the form $b - jc$, and $b + jc$, this gives

$$E_{L_2} = 2 E_1 \frac{L_2}{Z_1} \frac{a(b^2 + c^2)}{c^2 + (a - b)^2}\left[e^{-at} - e^{-bt}\left(\cos ct - \frac{a-b}{c} \sin ct\right)\right] \quad (36)$$

and

$$E_{L_1} = 2\,E_1 \frac{L_1}{Z_1} \frac{a(b^2 + c^2)}{c^2 + (a - b)^2} \Bigg\{ (1 + C_2\,L_2\,a^2)\,e^{-at}$$

$$- e^{-bt} \Bigg[\Bigg(1 + C_2\,L_2\,(2ab - b^2 - c^2) \Bigg) \cos ct$$

$$- \frac{a - b + C_2\,L_2 \{ b^2\,(a - b) - c^2\,(a + b) \}}{c} \sin ct \Bigg] \Bigg\} \qquad \textbf{(37)}$$

It has been found by Mr. J. E. Clem, who calculated the curves for this paper, that within the range where we apply the above equations the following simplifications are possible:

We have assumed L_1 and L_2 equal, and have therefore but one value of inductance as well as one of capacity. Dropping the subscripts, we have for equation (**18**)

$$x^3 + \frac{Z_1}{L}\,x^2 + \frac{2}{LC}\,x + \frac{Z_1}{L^2 C} = 0$$

Within the range of values used, $\frac{Z_1^2}{4L^2}$ is very small as compared with $\frac{2}{L\,C}$. The above equation may therefore, with sufficient accuracy, be written

$$x^3 + \frac{Z_1}{L}\,x^2 + \left(\frac{2}{LC} + \frac{Z_1^2}{4L^2} \right) x + \frac{Z_1}{L^2 C} = 0$$

This equation is factored, giving

$$\left(x + \frac{Z_1}{2L} \left(x^2 + \frac{Z_1}{2L}\,x + \frac{2}{LC} \right) = 0 \right.$$

The first factor gives

$$x = - \frac{Z_1}{2L}$$

and the second

$$x = -\frac{Z_1}{4L} \pm \sqrt{\frac{Z_1^2}{16\,L^2} - \frac{2}{CL}}$$

But $\frac{Z_1^2}{16\,L^2}$ may be neglected whence

$$x = -\frac{Z_1}{4L} \pm j\sqrt{\frac{2}{CL}}$$

The oscillatory case applies, with

$$a = \frac{Z_1}{2L},\ b = \frac{Z_1}{4L},\ \text{and } c = \sqrt{\frac{2}{CL}}$$

Substituting these values in equations (**36**) and (**37**), and remembering that a^2 and b^2 are small, as compared with c^2, it is found that these equations are represented with sufficient accuracy by the following:

$$E_{L_2} = E_1\left[e^{-\frac{Z_1}{2L}t} - e^{-\frac{Z_1}{4L}t}\left(\cos\sqrt{\frac{2}{LC}}\,t - \frac{Z_1}{4\sqrt{\frac{2L}{C}}}\sin\sqrt{\frac{2}{LC}}\,t\right)\right] \quad \textbf{(36a)}$$

and

$$E_{L_1} = E_1\left[e^{-\frac{Z_1}{2L}t} - e^{-\frac{Z_1}{4L}t}\left(\cos\sqrt{\frac{2}{LC}}\,t - \frac{5\,Z_1}{4\sqrt{\frac{2L}{C}}}\sin\sqrt{\frac{2}{LC}}\,t\right)\right] \quad \textbf{(37a)}$$

Written in the general form, these equations are

$$E_{L_2} = E_1\left[e^{-at} - e^{-bt}\left(\cos ct - \frac{b}{c}\sin ct\right)\right] \quad \textbf{(36b)}$$

and

$$E_{L_1} = E_1\left[e^{-at} + e^{-bt}\left(\cos ct - \frac{5b}{c}\sin ct\right)\right] \quad \textbf{(37b)}$$